Lecture 18

Expectation and Variance of a CRV

Recall that the expected value of a discrete random var X is $E[X] = \sum_x x\,P(X=x) = \sum_x x\,p(x)$, where $p(x) = P(X=x)$ is the pmf.

If X is a CRV, we have an analogous definition using the prob. density function:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$$

Ex: For $f(x) = \begin{cases} 0 & \text{for } x \le -1 \\ x-1 & \text{for } -1 \le x \le 0 \\ 1-x & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x \ge 1 \end{cases}$

What is the expected value?

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{0} x(x+1)\,dx + \int_{0}^{1} x(1-x)\,dx$$

$$= \int_{-1}^{0} x^2 + x\,dx + \int_{0}^{1} x - x^2\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^3}{3}\Big|_{0}^{1}$$

$$= -\left(\frac{-1}{3} + \frac{1}{2}\right) + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{3} - \frac{1}{2} + 1/2 - 1/3 = 0$$

Proposition: Let X be a random variable with $f(x)$ pdf and $g(X)$ a real-valued function of X. Then

$$E[g(X)] = \int g(x) f(x)\,dx.$$

See Prop 2.1 for a proof of essentially this fact.

Ex: Suppose X has pdf

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{o/w.} \end{cases}$$

Compute $E[e^X]$.

$$E[e^X] = \int_0^1 e^x f(x)\,dx = e^x \Big|_0^1 = e - 1$$

For a CRV, X, the variance, σ^2, is defined in the same way:

$$Var(X) = \sigma^2 = E[(X-\mu)^2]$$ proof is the same

$$= E[X^2] - E[X]$$ as the discrete case. See notes/text.

The Uniform Random Variable

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval. Then the <u>uniform random variable</u> on (α, β) is the variable X with pdf given by

$$f_X(x) = \begin{cases} \dfrac{1}{\beta - \alpha}. & x \in (\alpha, \beta) \\ 0 & o/w. \end{cases}$$

(This is the random variable that describes the game of choosing a random real number in the interval $(0,1)$).

Suppose that X is uniform on (α, β). What is $E[X]$?

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha}\, dx$$

$$= \frac{1}{(\beta - \alpha)} \frac{x^2}{2}\Big|_{\alpha}^{\beta} = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\cancel{(\beta - \alpha)}(\beta + \alpha)}{2\cancel{(\beta - \alpha)}}$$

$$= \beta + \alpha$$

$\frac{\quad}{2}$ ← midpoint of (α, β)

To compute the variance, first we find $E[X^2]$

$$E[X^2] = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}\,dx = \frac{1}{(\beta-\alpha)} \left.\frac{x^3}{3}\right|_\alpha^\beta$$

$$= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}.$$

So

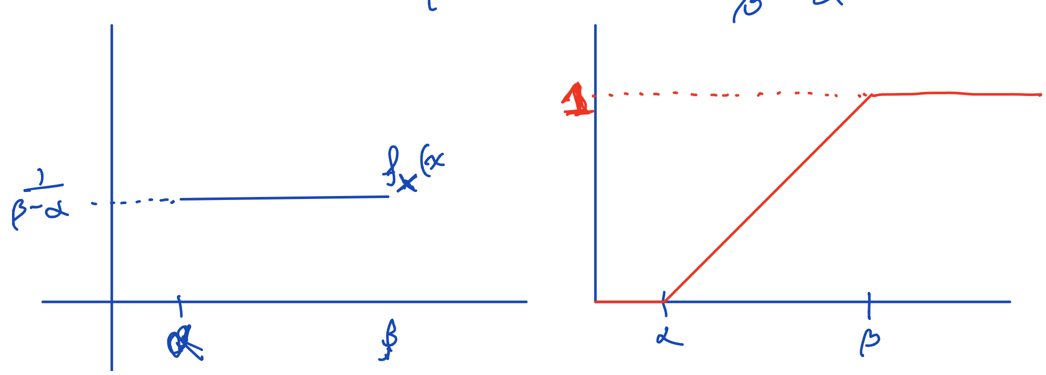
$$\mathrm{Var}(X) = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\alpha+\beta)^2}{4}.$$

$$= \frac{(\beta-\alpha)^2}{12}.$$

If X is uniform (α, β), then the cumulative distribution function is

$$F(x) = \int_a^x \frac{1}{\beta-\alpha}\,dt =$$

$$= \left.\frac{t}{\beta-\alpha}\right|_a^x = \frac{x-a}{\beta-\alpha}.$$

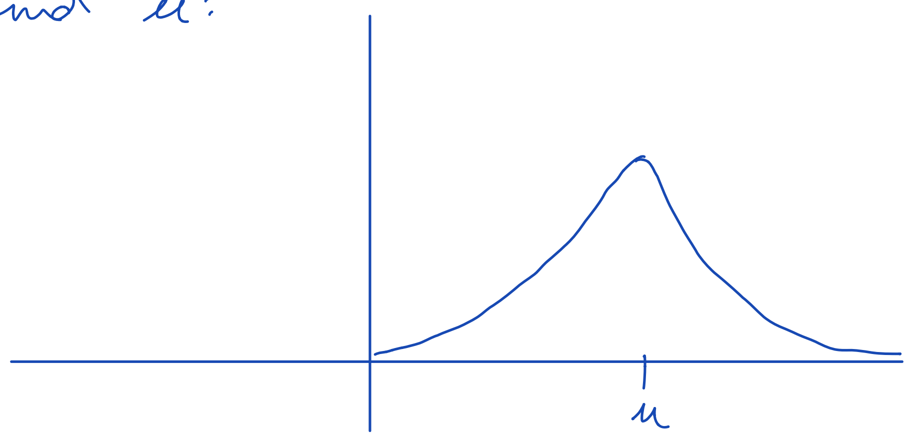


Normal Random Variables.

We say that X is normally distributed with mean μ and variance σ^2 (sometimes we write $X = N(\mu, \sigma^2)$) if X has pdf given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

The pdf is a bell-shaped function symmetric around μ:



- The normal distribution is the most important/commonly occurring distribution in all of probability.

- many natural phenomena are uniformly distributed.

One issue is that e^{-x^2} has no antiderivative..

Claim: $$\int_{-\infty}^{\infty} \underbrace{\frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}}_{f(x)} = 1.$$

Pf: substitute $y = \dfrac{(x-\mu)}{\sigma}$ to get.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

So it suffices to show that $\underbrace{\int_{-\infty}^{\infty} e^{-y^2/2}}_{I} = \sqrt{2\pi}$.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right).$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy.$$

polar substitution:
$x = r\cos\theta$
$y = r\cos\theta.$
$\Rightarrow x^2+y^2 = r^2.$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta.$$

Now $\dfrac{d}{dr}\left(e^{-r^2/2}\right) = -re^{-r^2/2}$

So $\int re^{-r^2/2}\,dr = -e^{-r^2/2}$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\lim_{t\to\infty} -e^{-t^2/2} + e^0\right)$$

$$= (2\pi)(1) = 2\pi$$

So $I^2 = 2\pi \Rightarrow I = \sqrt{2\pi}$ as required.